Quantal Symmetries in the Non-Linear Sigma Model With Maxwell–Chern–Simons Term[1]

Wang Yong-Long[2,3] and Li Zi-Ping[2]

The quantal symmetry property in the CP1 non-linear sigma model with Abelian–Maxwell–Chern–Simons (AMCS) term in $2 + 1$ dimensions is studied. In the Coulomb gauge, the system is quantized in the Faddeev–Senjanovic (FS) path-integral formalism. The canonical Ward identities for proper vertices under local gauge transformation are derived. Based on the quantal symmetries of a constrained Hamiltonian system, the fractional spin at the quantum level of this system is also presented as those of the system without Maxwell term.

KEY WORDS: constrained Hamiltonian system; fractional spin; CP1 non-linear sigma model.

1. INTRODUCTION

In $(2 + 1)$ dimensions space-time there exists the interesting possibility of fractional angular momentum and exotic statistics (Dowker, 1972; Laidlaw and DeWitt, 1971; Wu, 1984). It provides a realization of the fractional spin and statistics that is the $(2 + 1)$ dimensional O(3) nonlinear sigma model with a topological action (Bowick et al., 1986; Wilczek and Zee, 1983a; Wu and Zee, 1984). The model modified by the additions of Hopf term characterizing maps from S3 to S2 reveals the occurrence of fractional spin and statistics (Bowick *et al*., 1986 Mackenzie, 1988; Tsurumaru and Tsutsui, 1999; Wilczek and Zee, 1983b). The system can be cast in the form of a genuine gauge theory by the inclusion of the Chern-Simons (CS) term which implements fractional and statistics (Mukherjee, 1997; Panigrahi, *et al*., 1988). Recently, the O(3) non-linear sigma model with Hopf and CS terms is discussed in classical and quantum level (Banerjee, 1994; Li, 1996a), and the O(3) non-linear sigma model coupled to a topologically massive $U(1)$ gauge field including the Maxwell term (Karabali, 1987) is discussed. The

[2] College of Applied Science, Beijing University of Technology, Beijing, P. R. China.

[3] To whom correspondence should be addressed at College of Applied Science, Beijing University of Technology, Beijing 100022, P. R. China; e-mail: Wylong@emails.bjpu.edu.cn.

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fractional spin and statistics based on the canonical approach are always obtained through symmetric energy-momentum tensor not Noether's law. The CP1 nonlinear sigma model which is intimately related to the O(3) nonlinear sigma model in the long-range limit has received a lot of attention recently in connection with high-T_c superconductivity (Polyakov, 1988). Scalar (Jiang and Li, 1999) and spinor (Li and Li, 2002) QED with CS term are discussed, in which the fractional spin and statistics is revealed, too. We shall consider CP1 non-linear sigma model with AMCS term, and shall adopt path-integral quantization for this system to explain the existence of fractional spin rigorously at the quantum level. And the canonical Ward identities for proper vertices under local gauge transformation are derived.

2. FS PATH-INTEGRAL QUANTIZATION

We consider the gauged CP1 nonlinear σ model with Maxwell–Chern–Simons term in $(2+1)$ dimensions, and the Lagrangian of the system is given by (Karabali, 1987; Li, 1996a; Panigrahi $et\ al.$, 1988)

$$\mathsf{L} = \frac{1}{f}(D_\mu Z_k)^*(D^\mu Z_k) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi^2}\varepsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda \tag{1}$$

where f is coupling constant and assumed $f = 1$, $k = 1, 2$ and Z_k is a two-component complex field which satisfies the constraint

$$Z_k Z_k^* = |Z_1|^2 + |Z_2|^2 = 1 \tag{2}$$

and the covariant derivative $D_\mu = \partial_\mu - iA_\mu$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here A_μ is the CS gauge field. The canonical momentums conjugate to the fields Z_k, Z_k^*, and A_μ are

$$\pi_k = \frac{\partial \mathsf{L}}{\partial \dot{Z}_k} = (D_0 Z_k)^*, \bar{\pi}^k = \frac{\partial \mathsf{L}}{\partial \dot{Z}_k^*} = D_0 Z_k, \pi^i = \frac{\partial \mathsf{L}}{\partial \dot{A}_i}$$

$$= -F^{0i} + \frac{\theta}{2\pi^2}\varepsilon^{ij}A_j, \pi^0 = \frac{\partial \mathsf{L}}{\partial \dot{A}_0} = 0 \tag{3}$$

respectively. The primary constraints of the system are given by (Panigrahi $et\ al.$, 1988)

$$\Lambda^0 = \pi^0 \approx 0 \tag{4}$$

$$\theta^0 = Z_k Z_k^* - 1 \approx 0 \tag{5}$$

where symbol "$\approx$" means weak equality in Dirac sense. The canonical Hamiltonian density corresponding to the Lagrangian (1) is given by

$$\mathsf{H}_c = \pi^\mu \dot{A}_\mu + \pi^k \dot{Z}_k + \bar{\pi}^k \dot{Z}_k^* - \mathsf{L} = \mathsf{H}_0 + A_0\left[J^0 - \left(\partial_i \pi^i + \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_i A_j\right)\right] \tag{6}$$

with

$$H_0 = 2\pi^k \bar{\pi}^k - \frac{1}{2}\pi^i \pi^i - (D_\mu Z_k)^* (D^\mu Z_k)$$

$$+ \frac{1}{4}F_{ij}F^{ij} - \frac{\theta^2}{32\pi^4}A^i A_i + \frac{\theta}{2\pi^2}\varepsilon_{ij}\pi^i A^j \tag{7}$$

and $J_0 = i[(DZ_k)^* Z_k - (DZ_k)Z_k^*]$. The total Hamiltonian is given by

$$H_T = \int d^2x(\mathsf{H}_c + \lambda_0 \Lambda^0 + \omega_0 \theta^0) \tag{8}$$

The consistency condition $\{\Lambda^0, H_T\} \approx 0$ and $\{\theta^0, H_T\} \approx 0$ lead to secondary constraints

$$\Lambda^1 = 2J_0 - \left(\partial_i \pi^i + \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_i A_j\right) \approx 0 \tag{9}$$

$$\theta^1 = \pi^k Z_k + \bar{\pi}^k Z_k^* \approx 0 \tag{10}$$

respectively. The consistency of the secondary constraints (9) and (10) do not generate any new constraints. It is easy to check that the constraints (Λ^0, Λ^1) are first-class, and the others (θ^0, θ^1) are second-class. According to the theory of canonical quantization of constrained Hamiltonian system, for each first-class constraint a corresponding gauge condition should be chosen. We choose the Coulomb gauge $\Omega_0 = \partial_i A_i \approx 0$, the stationary of the Coulomb gauge $\partial_i \dot{A}_i \approx \{\Omega_0, H_T\} \approx 0$, and another gauge constraint is obtained

$$\Omega_1 = \nabla^2 A_0 + \partial_i \pi^i - \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_i A_j \tag{11}$$

According to FS path-integral quantization scheme, the phase-space generating functional of Green function for the system (1) is given by (Li and Jiang, 2002)

$$Z[J^\alpha, K_\alpha, U^l, V^n, W^i] = \int \mathsf{D}\phi_\alpha \mathsf{D}\pi^\alpha \mathsf{D}\lambda_l \mathsf{D}\mu^n \mathsf{D}\omega^i \delta(\Lambda)\delta(\Omega)\delta(\theta)\det|\{\Lambda^l, \Omega_n\}|$$

$$\times [\det|\{\theta_i, \theta_j\}|]^{\frac{1}{2}} \exp\left\{i\int d^3x(L^P + J^\alpha \phi_\alpha + K_\alpha \pi^\alpha\right.$$

$$\left. + U^l \lambda_l + V^n \mu_n + W^i \omega_i)\right\} \tag{12}$$

Here we have introduced the exterior sources $(J^\alpha, K_\alpha, U^l, V^n, W^i)$ with respect to the field $(\phi_\alpha, \pi^\alpha, \lambda_l, \mu_n, \omega_i)$, and the exterior sources $(K_k, \bar{K}_k, K_\mu)$ with respect to momenta $(\pi^k, \bar{\pi}^k, \pi^\mu)$, λ_l, μ_n, and ω_i are multiplier fields. It is easy to find out that $\det|\{\Lambda^k, \Omega_l\}|$ is independent of field variables. Thus, we can omit this factor

from the generating functional. Through the calculation of det $|\{\theta_i, \theta_j\}|$ and using the properties of δ-function and integral of Grassman variables, the expression (12) is written as

$$Z[J^\alpha, K_\alpha, U^i, V^i, W^i] = \int D\phi D\pi D\lambda D\mu D\omega \times \exp \left\{ i \int d^3x \left(L^P_{\text{eff}} \right. \right.$$

$$\left. \left. + J^\alpha \phi_\alpha + K_\alpha \pi^\alpha + U^i \lambda_i + V^i \mu_i + W^i \omega_i \right) \right\} \quad (13)$$

where

$$L^P_{\text{eff}} = L^P + L_m + L_{gh} \tag{14}$$

$$L^P = \pi_k \dot{Z}_k + \bar{\pi}_k \dot{Z}^*_k + \pi^\mu \dot{A}_\mu - H_c \tag{15}$$

$$L_m = \lambda_l \Lambda_l + \mu_n \Omega_n + \omega_i \theta_i \tag{16}$$

$$L_{gh} = 4\bar{C}(x)(Z_k(x) * Z^*_k(x))^2 C(x) \tag{17}$$

3. CANONICAL WARD IDENTITIES

Let us now construct the gauge transformation for a system with Lagrangian (1). The gauge generator for this system can be constructed via first-class constraints (4) and (9) (Li and Jiang, 2002)

$$G = \int_V d^2x [\dot{\varepsilon}(x)\Lambda^0 - \varepsilon(x)\Lambda^1]$$

$$= \int_V d^2x \left\{ \dot{\varepsilon}(x)\pi^0 - \varepsilon(x) \left[2J_0 - \left(\partial_i \pi^i + \frac{\theta}{2\pi^2} \varepsilon^{ij} \partial_i A_j \right) \right] \right\} \tag{18}$$

This generator produces the following transformation

$$\begin{cases} \delta Z_k = \{Z_k(x), G\} = -2i Z_k(x)\varepsilon(x) \\ \delta Z^*_k = \{Z^*_k(x), G\} = 2i Z^*_k(x)\varepsilon(x) \\ \delta A_\mu = \{A_\mu(x), G\} = -\partial_\mu \varepsilon(x) \\ \delta \pi^k = \{\pi^k(x), G\} = 2i\pi^k(x)\varepsilon(x) \\ \delta \bar{\pi}^k = \{\bar{\pi}^k(x), G\} = -2i\bar{\pi}^k(x)\varepsilon(x) \\ \delta \pi^\mu = \{\pi^\mu(x), G\} = \frac{\theta}{2\pi^2} \varepsilon^{ij} \partial_i \varepsilon(x) \end{cases} \tag{19}$$

Under the transformation (19) the canonical Lagrangian (15) is invariant. The change of (14) under the transformation (19) is given by

$$\delta L^P_{\text{eff}} = -\mu_0 \nabla^2 \varepsilon(x) + \mu_1(x) \left[\frac{\theta}{\pi^2} \varepsilon^{ij} \nabla^2 \varepsilon(x) - \nabla^2 \dot{\varepsilon}(x) \right] \tag{20}$$

The Jacobian of the transformation (19) is equal to unity. The generating functional (13) is invariant under the transformation (19); this yields the following Ward identities (Li, 1994, 1996)

$$
\left(-\nabla^2 \frac{\delta}{\delta U^0} + \nabla^2 \partial_0 \frac{\delta}{\delta U^1} + \frac{\theta}{\pi^2} \varepsilon^{ij} \nabla^2 \frac{\delta}{\delta U^1} - \partial_\mu J^\mu - 2\bar{J}^k \frac{\delta}{\delta \bar{J}^k} + 2J^k \frac{\delta}{\delta J^k} \right.
$$

$$
\left. + 2\bar{K}_k \frac{\delta}{\delta \bar{K}_k} - 2K_k \frac{\delta}{\delta K_k} - \frac{\theta}{2\pi^2} \varepsilon^{ij} \partial_i K_\mu \right) Z[J^\alpha, K_\alpha, U^l, V^n, W^i] = 0 \quad (21)
$$

Let $Z[J^\alpha, K_\alpha, U^l, V^n, W^i] = \exp\{iW[J^\alpha, K_\alpha, U^l, V^n, W^i]\}$ and use the definition of the generating functional of proper vertices $\Gamma[J^\alpha, K_\alpha, U^l, V^n, W^i]$ which is given by performing a functional Legendre transformation on $W[J^\alpha, K_\alpha, U^l, V^n, W^i]$,

$$
\Gamma[J^\alpha, K_\alpha, U^l, V^n, W^i] = W[J^\alpha, K_\alpha, U^l, V^n, W^i] - \int d^3x (J^\alpha \phi_\alpha
$$

$$
+ K_\alpha \pi^\alpha + U^l \lambda_l + V^n \mu_n + W^i \omega_i) \quad (22)
$$

and

$$
\begin{cases} \dfrac{\delta W}{\delta J^\alpha} = \phi_\alpha \\ \dfrac{\delta \Gamma}{\delta \phi_\alpha} = -J^\alpha \end{cases} \quad \begin{cases} \dfrac{\delta W}{\delta K_\alpha} = \pi^\alpha \\ \dfrac{\delta \Gamma}{\delta \pi^\alpha} = -K_\alpha \end{cases} \quad \begin{cases} \dfrac{\delta W}{\delta V^n} = \mu_n \\ \dfrac{\delta \Gamma}{\delta \mu_n} - V^n \end{cases} \quad (23)
$$

Then (22) becomes

$$
-\nabla^2 \mu_0 + \nabla^2 \dot{\mu}_1 + \frac{\theta}{\pi^2} \varepsilon^{ij} \nabla^2 \mu_1 + \partial_\mu \left(\frac{\delta \Gamma}{\delta A_\mu} \right) + 2Z_k \left(\frac{\delta \Gamma}{\delta Z_k} \right) - 2Z_k^* \left(\frac{\delta \Gamma}{\delta Z_k^*} \right)
$$

$$
-2\pi^k \left(\frac{\delta \Gamma}{\delta \pi^k} \right) + 2\bar{\pi}^k \left(\frac{\delta \Gamma}{\delta \bar{\pi}^k} \right) - \frac{\theta}{2\pi^2} \varepsilon^{ij} \partial_i \left(\frac{\delta \Gamma}{\delta \pi^\mu} \right) = 0 \quad (24)
$$

We functionally differentiate (24) with respect to $Z_k(x)$ and $Z_k^*(x)$, and set all fields (including multiplier fields) equated to zero, $A_\mu = Z_k = Z_k^* = \mu_0 = \mu_1 = \pi^k = \bar{\pi}^k = \pi^\mu = 0$; we obtain

$$
\partial_{x_1}^\mu \frac{\delta^3 \Gamma[0]}{\delta Z_k^*(x_3) \delta Z_k(x_2) \delta A_\mu(x_1)} + \frac{\theta}{2\pi^2} \varepsilon^{ij} \partial_{x_1}^i \frac{\delta^3 \Gamma[0]}{\delta Z_k^*(x_3) \delta Z_k(x_2) \delta \pi^\mu(x_1)}
$$

$$
= 2\delta(x_1 - x_3) \frac{\delta^2 \Gamma[0]}{\delta Z_k(x_2) \delta Z_k^*(x_1)} - 2\delta(x_1 - x_2) \frac{\delta^2 \Gamma[0]}{\delta Z_k^*(x_3) \delta Z_k(x_1)} \quad (25)
$$

Substituting (3) into (25), one gets

$$\partial^{\mu}_{x_1} \frac{\delta^3 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_2)\delta A_{\mu}(x_1)} = \delta(x_1 - x_3)\frac{\delta^2 \Gamma[0]}{\delta Z_k(x_2)\delta Z_k^*(x_1)}$$

$$-\delta(x_1 - x_2)\frac{\delta^2 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_1)} \tag{26}$$

Similarly, differentiating (24) many times with respect to field variables and setting all fields equal to zero, one can obtain various Ward identities for proper vertices.

4. FRACTIONAL SPIN AND STATISTICS

If the effective canonical action $I^P_{\text{eff}} = \int d^2x \mathsf{L}^P_{\text{eff}}$ is invariant under the following global transformation in extended phase space

$$\begin{cases} x^{\mu'} = x^{\mu} + \Delta x^{\mu} = x^{\mu} + \varepsilon_{\sigma}\tau^{\mu\sigma}(x, \varphi, \pi) \\ \varphi'(x') = \varphi(x) + \Delta\varphi(x) = \varphi(x) + \varepsilon_{\sigma}\xi^{\sigma}(x, \varphi, \pi) \\ \pi'(x') = \pi(x) + \Delta\pi(x) = \pi(x) + \varepsilon_{\sigma}\eta^{\sigma}(x, \varphi, \pi) \end{cases} \tag{27}$$

where φ and π denote: $\varphi = (Z_k, Z_k^*, A_{\mu}, \lambda_i, \mu_i, \omega_i)$, $\pi = (\pi^k, \bar{\pi}^k, \pi^{\mu})$, and $\varepsilon_{\sigma}(\sigma = 1, 2, \ldots, r)$ are infinitesimal arbitrary parameters, and the Jacobian of the transformation (27) is equal to unity, then, according to canonical Noether theorem in quantum formalism (Li, 1996a,b,c), there are conserved laws at the quantum level

$$Q^{\sigma} = \int_V d^3x [\pi(\xi^{\sigma} - \varphi_{,k}\tau^{k\sigma}) - \mathsf{H}_{\text{eff}}\tau^{0\sigma}] = \text{const} \tag{28}$$

where H_{eff} is an effective Hamiltonian density connected with $\mathsf{L}^P_{\text{eff}}$. Under the spatial rotation $\tau^{0\sigma} = 0$, A_{μ} are vector fields and the term L_{gh} does not involve the time derivative of field variables, and the Jacobian of the transformation of field variables are equal to zero. Thus, according to (28) the quantal conserved angular momentum for this system is given by

$$L = \int d^2x \varepsilon^{ij}\big[(x_i\pi^k\partial_j Z_k + x_i\bar{\pi}^k\partial_j Z_k^*) + \big(\pi_{\mu}S^{\mu\nu}_{ij} A_{\nu} + x_i\pi^{\mu}\partial_j A_{\mu}\big)\big] \tag{29}$$

where $S^{kl}_{ij} = \delta_i^k\delta_j^l - \delta_j^k\delta_i^l$. The quantal conserved angular momentum under the rotation in (x_1, x_2) plane coincides with the result derived from classical Noether theorem. Substituting (3) into (29), one gets

$$L = \int d^2x \varepsilon^{ij}(x_i\pi_k\partial_j Z_k + x_i\bar{\pi}_k\partial_j Z_k^* - x_i F_{0i}\partial_j A^i) - \int d^2x F_{0i'} S^{i'j'}_{ij} A'_j$$

$$+\frac{\theta}{2\pi^2}\varepsilon^{ij}\int d^2x x_i A_j\varepsilon^{kl}\partial_k A_l \tag{30}$$

On the gauge constrained surface, one has

$$\partial_i \pi^i = -\nabla^2 A_0 + \frac{\theta}{2\pi^2} \varepsilon^{ij} \partial_i A_j \tag{31}$$

Substituting (31) into (9), one gets

$$J_0 + \nabla^2 A_0 = \frac{\theta}{2\pi^2} \varepsilon^{ij} \partial_i A_j \tag{32}$$

From the Eq. (32), according to Gauss integral theorem we obtain

$$A_i(x) = -\frac{2\pi}{\theta} \varepsilon^{ij} \partial_x^j \int d^2 y\, G(x, y) J_0(y) \tag{33}$$

where

$$\Box G(x, y) = \delta^{(2)}(x - y) \tag{34}$$

Thus, the third term on the right hand of Eq. (19) is written as

$$\frac{\theta}{2\pi^2} \int d^2 x\, \varepsilon^{ij} x_i A_j \varepsilon^{kl} \partial_k A_l = \frac{\pi}{2\theta} Q^2 \tag{35}$$

where $Q = \int d^2 x\, J_0$, the first term on the right-hand of Eq. (20) is the orbit angular momentum operator, the second is normal spin term, the third is the anomalous one which is interpreted as an anomalous spin operator (Li, 1996a). We denote this spin operator by S, $S = \pi Q^2/2\theta$, and the one-particle (anyon) state is denoted by $|1>_{\text{any}}$, which carry one unit of charge. Then, if one rotates the one-particle state with S, one obtains

$$e^{i\beta S}|1\rangle_{\text{any}} = e^{i\beta(\pi/2\theta)}|1\rangle_{\text{any}} \tag{36}$$

where β is the rotation parameter. The eigenvalue of spin operator S is the spin s. Thus one obtains a relation between the spin s and the coefficient θ in CS term, namely

$$s = \frac{\pi}{2\theta} \tag{37}$$

If we take β as 2π, for $\theta = \pi/(2n + 1)(n \in Z)$, the one-particle state picks up a minus sign implying it is a fermionic, and these values of θ let the spin s take half-integer values. While, for $\theta = \pi/(2\pi)(n \in Z)$, the one-particle state does not change, and hence it becomes bosonic, and the spin s takes integer values, for the other values of θ, the state becomes anionic, and the spin s is fractional. The fractional spin of the CP1 non-linear sigma model with AMCS term is also occurrence as well as the Maxwell kinetic term is absent (Panigrahi et al., 1988). It is worthwhile to point out that we do not find out the difference of the fractional spin term as those between in (Jiang et al., 2004) and (Kim et al., 1994).

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